Analysis of Non-Local Euclidean Medians and Its Improvement

Zhonggui Sun and Songcan Chen

***Abstract*—Non-Local Euclidean Medians (NLEM) has recently been proposed and shows more effective than Non-Local Means (NLM) in removing heavy noise. In this letter, we find the inconsistency between the two dissimilarity measures in NLEM can affect its robustness, thus develop an improved version (INLEM) to compensate such an inconsistency. Further, we provide a concise convergence proof for the iterative algorithm used in both NLEM and INLEM. Finally, our experiments on synthetic and natural images show that INLEM achieves encouraging results.**

***Index Terms*—Improved non-local Euclidean medians (INLEM), image denoising, non-local Euclidean medians (NLEM), non-local means (NLM).**

I. Introduction

NON-LOCAL denoising methods have drawn a lot of attention in the image processing community and chiefly originate from the non-local means (NLM) proposed by Buades *et al.* [1]. Unlike local counterparts which typically operate within a local neighborhood, NLM computes weighted means in a non-local way by employing the between-patch dissimilarity (measure). Despite simple in idea, NLM outperforms some popular filters [4] and then motivates many successors proposed. Such a patch-based and non-local viewpoint has become a core of most state-of-the-art filters including BM3D [2], K-SVD [3] *et al.*, as reviewed in [4].

Along the line, recently, Chaudhury *et al.* proposed the Non-local Euclidean medians (NLEM) to improve robust performance of NLM to heavy (large noise level) noise by replacing the Euclidean mean with Euclidean median [5]. Unlike those weighted mean followers of NLM, NLEM is its weighted median variant and inherits robustness of the median filters to outlier or heavy noise [6]. However, we find that in its implementation, NLEM adopts two different kinds of between-patch measures: one is the Euclidean norm in the definition of the Euclidean median and the other is its squares in the definition of weight computation. In fact, measures are often task-dependent [7] and yield different robustness. Thus in NLEM, such a joint use of inconsistent measures likely discounts its robustness in denoising (as confirmed in our experiments). The observation motivates us to propose an improved NLEM (INLEM) to eliminate such an inconsistency. Overall, our main contributions can be summarized as follows:

1) We point out and analyze the inconsistency of the two kinds of between-patch measures used in NLEM.
2) We develop a new non-local median filter (INLEM) with consistent measures and obtain encouraging denoising effect.
3) For the iterative algorithm of both NLEM and INLEM, we give a quite concise convergence proof.

The rest of this letter is structured as follows: In the next section, we briefly review the related works including NLM and NLEM. Then in Section III, we develop INLEM and give our convergence proof for the iterative algorithm. In Sections IV and V, we provide experiments to demonstrate the advantages of our filter and a brief conclusion, respectively.

Manuscript received October 26, 2012; revised January 07, 2013; accepted January 30, 2013. Date of publication February 05, 2013; date of current version February 14, 2013. This work was supported in part by the National Natural Science Foundation of China under Grants 61170151 and 61035003 and also by the Qing Lan Project. The associate editor coordinating the review of this manuscript and approving it for publication was Prof. Yiannis Andreopoulos.

Z. Sun is with College of Computer Science and Technology (CCST), Nanjing University of Aeronautics & Astronautics (NUAA), 210016 Nanjing, China, and also with the Department of Mathematics Science, Liaocheng University, 252000 Liaocheng, China (e-mail: altlp@nuaa.edu.cn).

S. Chen is with College of Computer Science and Technology (CCST), Nanjing University of Aeronautics & Astronautics (NUAA), 210016 Nanjing, China (e-mail: s.chen@nuaa.edu.cn).

Digital Object Identifier 10.1109/LSP.2013.2245322

II. Related Works

A. Non-Local Means (NLM)

Like literatures [1]–[5], we also focus on the Gaussian noise with mean zero and variance σ^2. Suppose Y is an observed (noisy) image, X is its corresponding noise-free image to be recovered. Let $Y(i)$ denote the gray value of pixel$i$ in Y, Y_j the image patch centered at pixel j in Y. $X(i)$ and X_j are similarly defined. In NLM [1], $X(i)$ is estimated as a weighted mean of other pixels in a non-local search window in Y and the weight between pixels i and j is defined by the similarity between patches Y_i and Y_j. Algorithm 1 describes the implementation flowchart of NLM.

Algorithm 1 Non-Local Means (NLM)

Input: Noisy image Y and paramers h, s, k.

Output: Denoised image $\hat{X}$.

Step 1: Extract patch Y_i with radius k around every pixel i.

Step 2: For every pixel i, do

a) Set $w_{ij} = \exp((-\|Y_i - Y_j\|^2)/(h^2))$ for every $j \in S_i$.

b) Find$\hat{X}_i = \arg\min_{X_i} \sum_{j\in S_i} w_{ij}\|X_i - Y_j\|^2$.

c) Assign$\hat{X}(i)$ the value of the center pixel in $\hat{X}_i$.

where S_i is the search window centered at pixel i, s and k denote the radii of the window and patch respectively. $\|\cdot\|$ is the Euclidean norm.

Note that Euclidean mean (or mean for short) [6] is actually a closed-form solution to Step 2(b). Thus, operating Step 2(b)–(c) can lead to the following (closed-form) weighted mean:

$$\hat{X}(i)=\frac{\sum_{j\in S_i} w_{ij}Y(j)}{\sum_{j\in S_i} w_{ij}}, \tag{1}$$

which is just the mean of NLM proposed in [1].

Although NLM has been proved to achieve state-of-the-art denoising effect in most cases [4], the weights w_{ij} involved are all based on noisy image and thus naturally affect final performance especially in heavy noise, which motivated Chaudhury *et al.* to propose NLEM [5].

B. Non-Local Euclidean Medians (NLEM)

Aiming to enhance the denoising ability of NLM for heavy noise, more recently, Chaudhury *et al.* proposed NLEM filter [5]. Only difference from NLM, instead of computing the Euclidean mean in Step 2(b), is that NLEM computes a minimizer by optimizing Step $2(\mathrm{b}')$ defined as follows:

$$\mathrm{Step}\ 2(b')\quad \hat{X}_i=\underset{X_i}{\operatorname{argmin}}\sum\nolimits_{j\in S_i} w_{ij}\|X_i-Y_j\| \tag{2}$$

while all the other steps of NLEM keep unchanged.

Reference [5] has shown that the minimizer in (2) is more robust than the Euclidean mean derived from Step 2(b) and is called the Euclidean median (or median here for short) [6]. It is such a median that NLEM obtains stronger robusness to heavy noise than NLM [5]. Among many existing methods of computing the median, the Weiszfeld algorithm [8], [9] is deemed as the simplest and comonly-used one [5]. Concretely, the algorithm starts from a current estimate $\hat{X}_i^{(t)}$ to seek the next iterative solution by optimizating the following:

$$\hat{X}_i^{(t+1)}=\underset{X_i}{\operatorname{argmin}}\sum_{j\in S_i} w_{ij}\frac{\|X_i-Y_j\|^2}{\left\|\hat{X}_i^{(t)}-Y_j\right\|}, \tag{3}$$

Actually this is exactly an iteratively reweighted least squares (IRLS) problem which yields the following result

$$\hat{X}_i^{(t+1)}=\frac{\sum\limits_{j\in S_i}\frac{w_{ij}}{\left\|\hat{X}_i^{(t)}-Y_j\right\|}Y_j}{\sum\limits_{j\in S_i}\frac{w_{ij}}{\left\|\hat{X}_i^{(t)}-Y_j\right\|}}. \tag{4}$$

To avoid $\|\hat{X}_i^{(t)}-Y_j\|=0$ and speedup the iterations, Chaudhury *et al.* borrow the recent idea from [10], [11] to regularize (4) by adding a small positive bias ε and form the regularized iteration (5):

$$\hat{X}_i^{(t+1)}=\frac{\sum\limits_{j\in S_i}\frac{w_{ij}}{\left(\left\|\hat{X}_i^{(t)}-Y_j\right\|^2+\varepsilon_t^2\right)^{1/2}}Y_j}{\sum\limits_{j\in S_i}\frac{w_{ij}}{\left(\left\|\hat{X}_i^{(t)}-Y_j\right\|^2+\varepsilon_t^2\right)^{1/2}}}, \tag{5}$$

where ε_t tends to 0 in monotonous decreasing way. Compared with the Weiszfeld algorithm, [5] has empirically shown that the iterative algorithm (5) gets faster convergence, typically just 3 to 4 steps.

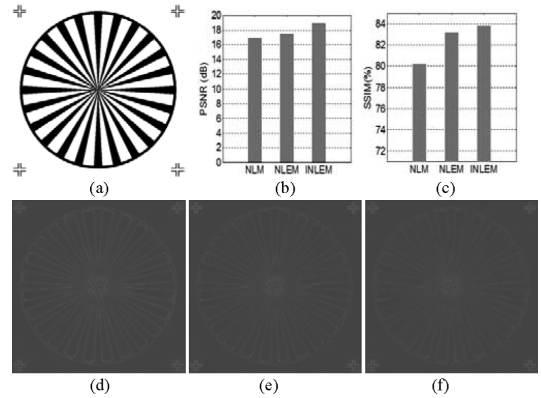

Fig. 1. The denoising effects on image "Pat" at $\sigma=50$: (a) Original image; (b) PSNR values; (c) SSIM values; The method noise for (d) NLM, (e) NLEM and (f) INLEM respectively.

As a median variant of NLM, NLEM obtains better performance in removing heavy noise.

III. Improved Non-Local Euclidean Medians (INLEM)

A. Motivation of Algorithm

By using the Euclidean norm to replace its squares in Step $2(\mathrm{b}')$, NLEM obtains stronger robustness. However, similar to Step 2(a) in NLM, the weights in NLEM are still based on the squared Euclidean norm. In fact, according to [7], measures are often task-dependent and different measures can yield robustness to different degrees. Thus such measure inconsistency in NLEM could result in a possibly-discounted effect in denoising heavy noise, which motivates us attempt to mitigate such an adverse effect by removing the inconsistency. A naive idea is using the Euclidean norm to replace its squares in computing the weights. That is, Step 2 (a) of Algorithm 1 (please note that this step is the same in both NLM and NLEM) is substituted by

$$\mathrm{Step}\ 2(a')\quad w_{ij}=\exp\left(-\frac{\|Y_i-Y_j\|}{h}\right). \tag{6}$$

Now we call this measure-consistent version of NLEM as INLEM. In such two non-local median filters, except that Step $2(\mathrm{a}')$ is different, the rest steps are the same.

To illustrate feasibility of INLEM, we conduct a preliminary denoising test on the synthetic image "Pat" which is generated by running command *imread ('testpat1.png')* in Matlab 7.11 (MathWorks Co.). We first add noise with $\sigma=50$ to the image and then compare the corresponding denoising effects of NLM, NLEM and INLEM (more experiments and specific parameter settings will be provided in Section 4).

Three frequently-used evaluation indexes of measuring denoising effect, including PSNR, Structural Similarity (SSIM) [12] and the method noise [1], [4], are respectively shown in Fig. 1. From both PSNR and SSIM, it can be found that INLEM obtains the best effect among the three filters involved in this condition. Meanwhile, the method noise for INLEM displays the least structural features visually, meaning that INLEM has relatively stronger ability in preserving details.

Before carrying out more comprehensive comparative experiments, let us elaborate rationality of consistent measures from an optimization objective viewpoint.

B. Theoretical Analysis of Consistency

In terms of the non-local denoising framework proposed in [13], [14], a non-local filtering model can be formulated as the following optimization problem:

$$\min_{W,X_i} \sum_{j\in S_i} w_{ij}\|X_i - Y_j\|^p + \lambda \sum_{j\in S_i} w_{ij}\log w_{ij}$$
$$\text{s.t.} \quad w_{ij} \geq 0, \quad \sum_{j\in S_i} w_{ij} = 1. \tag{7}$$

where $W = \{w_{ij} | j \in S_i\}$ is a set of the weights, $p > 0$ is a constant, $\sum_{j\in S_i} w_{ij}\|X_i - Y_j\|^p$ and $\sum_{j\in S_i} w_{ij}\log w_{ij}$ are data-dependent term and regularization term (based on Shannon entropy) respectively. $\lambda \geq 0$ is a trade-off parameter between the two terms.

To solve the problem (7), we initially set $X_i = Y_i$ to solve w_{ij} and then fix w_{ij} to solve X_i. Consequently, the solution to the problem can be conducted in two following steps S1–2:

$$\text{S1} \quad w_{ij} = \exp\left(-\frac{\|Y_i - Y_j\|^p}{\lambda}\right), \tag{8}$$

$$\text{S2} \quad \min_{X_i} \sum_{j\in S_i} w_{ij}\|X_i - Y_j\|^p. \tag{9}$$

Now it can be observed that the measures in the above two steps are consistent and both defined by $\|\cdot\|^p$. In particular, when set $p = 2$ and $\lambda = h^2$, the above two-steps respectively turn back to Step 2(a) and (b) in Algorithm 1. Thus NLM obtains an interpretation in terms of such a non-local filtering model. However, an inconsistency will be invoked when using the same model to interpret NLEM: on one hand, NLEM adopts the squared Euclidean norm to compute the weights, i.e., $p = 2$ in (8); on the other hand, NLEM adopts Step $2(\mathrm{b}')$ in (2) to compute the median, i.e., $p = 1$ in (9). It is very inconsistency that makes it possible to further improve NLEM in denoising.

To derive a more robust non-local median filter with consistent measures, we set $p = 1$ and $\lambda = h$ in (7), then get the corresponding solution in terms of (8) and (9) which exactly corresponds to Step $2(\mathrm{a}')$ in (6) and Step $2(\mathrm{b}')$ in (2) respectively. That is, the newly-obtained optimization result just equals to INLEM, which coincides with our naive intuition.

C. Convergence Proof

Equation (5) is used both in NLEM and INLEM to iteratively solve (2) for the Euclidean median. And a related convergence proof has been shown in [11] in a more elaborate way. However, as stated in [5], the iterative algorithm discussed in [11] is "a particular flavor" of (5). In this letter, we will provide a quite concise convergence proof for (5) and summarize in Theorem I.

Theorem I: The objective value in (2) always monotonically decreases in terms of iteration (5) until convergence.

Proof: For convenience, suppose that all image patches in this formulation are represented as corresponding row vectors. This way, (5) can equivalently be reformulated as

$$\hat{X}_i^{(t+1)} = \operatorname*{argmin}_{X_i} \sum_{j\in S_i} w_{ij} \frac{\|X_i - Y_j\|^2}{\left(\left\|\hat{X}_i^{(t)} - Y_j\right\|^2 + \varepsilon_t^2\right)^{\frac{1}{2}}}$$

Then, according to the optimality of $\hat{X}_i^{(t+1)}$, we have

$$\begin{aligned}
&\sum_{j\in S_i} \frac{w_{ij}\left\|\hat{X}_i^{(t+1)} - Y_j\right\|^2}{2\left(\left\|\hat{X}_i^{(t)} - Y_j\right\|^2 + \varepsilon_t^2\right)^{\frac{1}{2}}} \\
&\leq \sum_{j\in S_i} \frac{w_{ij}\left\|\hat{X}_i^{(t)} - Y_j\right\|^2}{2\left(\left\|\hat{X}_i^{(t)} - Y_j\right\|^2 + \varepsilon_t^2\right)^{\frac{1}{2}}} \\
&\Rightarrow \sum_{j\in S_i} \frac{\left\|w_{ij}\left[\hat{X}_i^{(t+1)} - Y_j, \varepsilon_t\right]\right\|^2}{2\left\|w_{ij}\left[\hat{X}_i^{(t)} - Y_j, \varepsilon_t\right]\right\|} \\
&\leq \sum_{j\in S_i} \frac{\left\|w_{ij}\left[\hat{X}_i^{(t)} - Y_j, \varepsilon_t\right]\right\|^2}{2\left\|w_{ij}\left[\hat{X}_i^{(t)} - Y_j, \varepsilon_t\right]\right\|} \\
&\Rightarrow \sum_{j\in S_i} \left\|w_{ij}\left[\hat{X}_i^{(t+1)} - Y_j, \varepsilon_t\right]\right\| \\
&\quad - \left(\sum_{j\in S_i} \left\|w_{ij}\left[\hat{X}_i^{(t+1)} - Y_j, \varepsilon_t\right]\right\| - \sum_{j\in S_i} \frac{\left\|w_{ij}\left[\hat{X}_i^{(t+1)} - Y_j, \varepsilon_t\right]\right\|^2}{2\left\|w_{ij}\left[\hat{X}_i^{(t)} - Y_j, \varepsilon_t\right]\right\|}\right) \\
&\leq \sum_{j\in S_i} \left\|w_{ij}\left[\hat{X}_i^{(t)} - Y_j, \varepsilon_t\right]\right\| \\
&\quad - \left(\sum_{j\in S_i} \left\|w_{ij}\left[\hat{X}_i^{(t)} - Y_j, \varepsilon_t\right]\right\| - \sum_{j\in S_i} \frac{\left\|w_{ij}\left[\hat{X}_i^{(t)} - Y_j, \varepsilon_t\right]\right\|^2}{2\left\|w_{ij}\left[\hat{X}_i^{(t)} - Y_j, \varepsilon_t\right]\right\|}\right)
\end{aligned}$$

[15] has shown that for any nonzero vectors $U_i^{(t)}|_{i=1}^r$

$$\sum_{i=1}^r U_i^{(t+1)} - \sum_{i=1}^r \frac{\left\|U_i^{(t+1)}\right\|^2}{2\left\|U_i^{(t)}\right\|} \leq \sum_{i=1}^r U_i^{(t)} - \sum_{i=1}^r \frac{\left\|U_i^{(t)}\right\|^2}{2\left\|U_i^{(t)}\right\|}$$

where r is an arbitrary number. Thus we have

$$\begin{aligned}
&\sum_{j\in S_i} \left\|w_{ij}\left[\hat{X}_i^{(t+1)} - Y_j, \varepsilon_t\right]\right\| \\
&\leq \sum_{j\in S_i} \left\|w_{ij}\left[\hat{X}_i^{(t)} - Y_j, \varepsilon_t\right]\right\| \\
&\Rightarrow \sum_{j\in S_i} w_{ij}\left\|\hat{X}_i^{(t+1)} - Y_j\right\| \\
&\leq \sum_{j\in S_i} w_{ij}\left\|\hat{X}_i^{(t)} - Y_j\right\|
\end{aligned}$$

TABLE I
COMPARISON OF NLM, NLEM AND INLEM IN TERMS OF PSNR AND SSIM, AT NOISES LEVELS $\sigma = 10, 20, \ldots, 100$

Image	Method	PSNR (dB)									
	NLM	**33.31**	27.99	23.69	20.06	17.43	15.18	13.30	11.86	10.79	10.02
Pat	NLEM	33.07	**28.02**	**24.30**	20.92	18.45	16.37	14.49	12.93	11.69	10.74
	INLEM	32.04	26.87	24.13	**21.42**	**18.82**	**16.69**	**14.95**	**13.55**	**12.42**	**11.50**
	NLM	**32.69**	27.82	25.40	23.77	22.55	21.60	20.86	20.27	19.78	19.38
Lena	NLEM	32.36	28.00	25.66	24.10	22.94	22.01	21.24	20.61	20.07	19.62
	INLEM	31.84	**28.25**	**26.00**	**24.39**	**23.26**	**22.36**	**21.61**	**20.96**	**20.40**	**19.91**
	NLM	**30.16**	25.40	23.38	22.29	21.48	20.82	20.30	19.87	19.51	19.20
Boats	NLEM	29.99	25.52	23.42	22.42	21.69	21.07	20.54	20.09	19.70	19.38
	INLEM	30.10	**26.08**	**23.90**	**22.74**	**21.98**	**21.37**	**20.84**	**20.38**	**19.98**	**19.62**
	NLM	28.27	23.49	21.76	20.78	20.10	19.58	19.14	18.75	18.42	18.11
Bridge	NLEM	28.39	23.63	21.87	20.91	20.25	19.75	19.34	18.96	18.62	18.30
	INLEM	**28.85**	**24.20**	**22.34**	**21.26**	**20.57**	**20.06**	**19.63**	**19.24**	**18.88**	**18.54**
		SSIM (%)									
	NLM	**96.24**	93.64	90.72	86.15	80.21	72.65	64.19	56.01	48.97	43.33
Pat	NLEM	96.13	**93.68**	**91.12**	87.57	83.15	77.50	70.54	62.95	55.61	49.12
	INLEM	95.53	92.68	90.77	**88.00**	**83.76**	**78.37**	**72.29**	**66.03**	**59.96**	**54.33**
	NLM	89.76	80.53	74.30	69.03	64.41	60.58	57.56	55.26	53.51	52.20
Lena	NLEM	89.79	80.81	74.97	70.18	**65.93**	**62.21**	**59.11**	**56.61**	**54.62**	**53.10**
	INLEM	**89.92**	**82.10**	**75.75**	**70.41**	65.85	61.98	58.76	56.14	54.04	52.38
	NLM	82.33	68.07	59.87	55.01	51.48	48.68	46.50	44.80	43.49	42.49
Boats	NLEM	82.43	68.45	60.14	55.49	52.15	49.47	47.28	45.50	44.07	42.97
	INLEM	**84.42**	**71.81**	**62.99**	**57.50**	**53.59**	**50.48**	**47.97**	**45.94**	**44.30**	**43.00**
	NLM	79.10	54.62	44.76	38.98	35.05	32.14	29.95	28.23	26.88	25.83
Bridge	NLEM	80.91	55.32	45.29	39.58	35.67	32.80	30.61	28.85	27.46	26.32
	INLEM	**85.58**	**61.62**	**50.19**	**43.66**	**39.31**	**36.10**	**33.58**	**31.53**	**29.84**	**28.44**

which indicates that the objective value

$$\sum_{j \in S_i} w_{ij} \| X_i^{(t+1)} - Y_j \|$$

always monotonically decreases until convergence to an optimal $\hat{X}_i$ through (5).

IV. EXPERIMENTS

In our experiments, we compare the denoising effects of NLM, NLEM and INLEM on the synthetic image "Pat" and three benchmark natural images. In the three natural ones, "Lena" is homogeneous, "Bridge" contains many details [16] and "Boats" has the detail degree between the above two images. The four images involved are all with the size of 256×256. And the grey values of all images, including the original and their noisy versions, are all limited to the interval [0, 255].

For NLM and NLEM, we set $s = 10$, $k = 3$ and $h = 10\sigma$ respectively to the values recommended in [1], [5]. For INLEM, we set $h = 4\sigma$, s and k are taken as the same values as those in the other two filters involved, respectively. Obviously such parameter setting for INLEM is not necessarily the best, however the resulting experimental results show still promising.

For comparing robustness, we add noise to the test images with the same parameter setting as that in [5], i.e., from $\sigma = 10$ to $\sigma = 100$ with the increment of 10. The PSNR and SSIM [12] values of the corresponding results are reported in Table I.

It can be found from Table I that:

1) In terms of PSNR, INLEM achieves more robust effects than NLEM in removing heavy noise. Particularly when $\sigma > 30$, INLEM consistently possesses the highest PSNR values among the three filters involved on all the test images.
2) In terms of SSIM, INLEM also shows similar performance except on image "Lena" where it is slightly poorer than NLEM but better than NLM for heavy noise.
3) On image "Bridge" with more details, INLEM performs the best no matter whether noise is light or heavy, which further demonstrates its ability in preserving details.

V. CONCLUSION

In this letter, by unifying the measures involved in NLEM, we develop a new non-local median filter (INLEM) to further promote denoising effects especially for heavy noise. Moreover, we provide a concise convergence proof for the iterative algorithm used in both NLEM and INLEM.

As a non-local filter, INLEM shares high computational cost with both NLM and NLEM. However, fortunately, some speeding-up algorithms for NLM, like [17], [18], have recently been designed. These strategies can likewise be borrowed to speedup INLEM including NLEM.

Similar setting to [5], a wide range of noise level (σ from 10 to 100) is used in the experiments here to validate the ability of INLEM in removing heavy Gaussian noise. However, it should be noted that Gaussian noise will assume similar properties of impulse noise when the variance is too large. Moreover, for the extreme case, a median-based method can likely more powerful. Thus adapting INLEM to this extreme condition is meaningful and will be a future work.

ACKNOWLEDGMENT

The authors would like to thank Dr. Kunal N. Chaudhury for generously sharing his codes and giving valuable advice.

REFERENCES

[1] A. Buades, B. Coll, and J. Morel, "A review of image denoising algorithms, with a new one," *SIAM Interdisc. J.: Multiscale Model. Simul.*, vol. 4, no. 2, pp. 490–530, 2005.

[2] K. Dabov *et al.*, "Image denoising by sparse 3-D transform-domain collaborative filtering," *IEEE Trans. Image Process.*, vol. 16, pp. 2080–2095, 2007.

[3] M. Aharon, M. Elad, and A. M. Bruckstein, "The K-SVD: An algorithm for designing of overcomplete dictionaries for sparse representation," *IEEE Trans. Signal Process.*, vol. 54, no. 11, pp. 4311–4322, 2006.

[4] A. Buades, B. Coll, and J. Morel, "Image denoising methods. A new nonlocal principle," *SIAM Rev.: Multiscale Model. Simul.*, vol. 52, no. 1, pp. 113–147, 2010.

[5] K. N. Chaudhury and A. Singer, "Non-local Euclidean medians," *IEEE Signal Process. Lett.*, vol. 19, no. 11, pp. 745–748, 2012.

[6] P. J. Huber and E. M. Ronchetti, *Robust Statistics*, 2nd ed. Hoboken, NJ, USA: Wiley, 2009.

[7] S. Cha, "Comprehensive survey on distance/similarity measures between probability density functions," *Int. J. Math. Models Meth. Appl. Sci.*, vol. 1, no. 2, pp. 300–307, 2007.

[8] E. Weiszfeld, "Sur le point pour lequel la somme des distances de n points donnes est minimum," *Tohoku Math. J.*, vol. 43, pp. 355–386, 1937.

[9] G. Xue and Y. Ye, "An efficient algorithm for minimizing a sum of euclidean norms with applications," *SIAM J. Optim.*, vol. 7, no. 4, pp. 1017–1036, 1997.

[10] R. Chartrand and Y. Wotao, "Iteratively reweighted algorithms for compressive sensing," in *Proc. ICASSP 2008*, 2008, pp. 3869–3872.

[11] I. Daubechies *et al.*, "Iteratively reweighted least squares minimization for sparse recovery," *Commun. Pure Appl. Math.*, vol. 63, no. 1, pp. 1–38, 2010.

[12] W. Zhou *et al.*, "Image quality assessment: From error visibility to structural similarity," *IEEE Trans. Image Process.*, vol. 13, no. 4, pp. 600–612, 2004.

[13] G. Facciolo *et al.*, , D. Cremers, Ed. *et al.*, "Exemplar-based interpolation of sparsely sampled images," in *Energy Minimization Methods in Computer Vision and Pattern Recognition*. Berlin/Heidelberg, Germany: Springer, 2009, pp. 331–344, Lecture Notes in Computer Science.

[14] P. Arias, V. Caselles, and G. Sapiro, , D. Cremers, Ed., "A variational framework for non-local image inpainting," in *Energy Minimization Methods in Computer Vision and Pattern Recognition*. Berlin/Heidelberg, Germany: Springer, 2009, pp. 345–358, Lecture Notes in Computer Science.

[15] F. Nie *et al.*, "Efficient and robust feature selection via joint l2, 1-norms minimization," *Adv. Neural Inf. Process. Syst. (NIPS)*, vol. 23, pp. 1813–1821, 2010.

[16] R. Garnett *et al.*, "A universal noise removal algorithm with an impulse detector," *IEEE Trans. Image Process.*, vol. 14, no. 11, pp. 1747–1754, Nov. 2005.

[17] R. Vignesh, B. T. Oh, and C. C. J. Kuo, "Fast non-local means (NLM) computation with probabilistic early termination," *IEEE Signal Process. Lett.*, vol. 17, no. 3, pp. 277–280, Mar. 2010.

[18] N. Dowson and O. Salvado, "Hashed non-local means for rapid image filtering," *IEEE Trans. Patt. Anal. Mach. Intell.*, vol. 33, no. 3, pp. 485–499, 2011.