Efficient $\ell_1/\ell_q$ Norm Regularization

Jun Liu  Jieping Ye

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Abstract

Sparse learning has recently received increasing attention in many areas including machine learning, statistics, and applied mathematics. The mixed-norm regularization based on the $\ell_1/\ell_q$ norm with $q > 1$ is attractive in many applications of regression and classification in that it facilitates group sparsity in the model. The resulting optimization problem is, however, challenging to solve due to the structure of the $\ell_1/\ell_q$-regularization. Existing work deals with special cases including $q = 2, \infty$, and they can not be easily extended to the general case. In this paper, we propose an efficient algorithm based on the accelerated gradient method for solving the $\ell_1/\ell_q$-regularized problem, which is applicable for all values of $q$ larger than 1, thus significantly extending existing work. One key building block of the proposed algorithm is the $\ell_1/\ell_q$-regularized Euclidean projection (EP$_{1q}$). Our theoretical analysis reveals the key properties of EP$_{1q}$ and illustrates why EP$_{1q}$ for the general $q$ is significantly more challenging to solve than the special cases. Based on our theoretical analysis, we develop an efficient algorithm for EP$_{1q}$ by solving two zero finding problems. Experimental results demonstrate the efficiency of the proposed algorithm.

1 Introduction

Regularization has played a central role in many machine learning algorithms. The $\ell_1$-regularization has recently received increasing attention, due to its sparsity-inducing property, convenient convexity, strong theoretical guarantees, and great empirical success in various applications. A well-known application of the $\ell_1$-regularization is the Lasso [32]. Recent studies in areas such as machine learning, statistics, and applied mathematics have witnessed growing interests in extending the $\ell_1$-regularization to the $\ell_1/\ell_q$-regularization [2, 7, 14, 23, 29, 37, 38]. This leads to the following $\ell_1/\ell_q$-regularized minimization problem:

$$\min_{W \in \mathbb{R}^p} f(W) = l(W) + \lambda \varpi(W),$$  \hspace{1cm} (1)$$

where $W \in \mathbb{R}^p$ denotes the model parameters, $l(\cdot)$ is a convex loss dependent on the training samples and their corresponding responses, $W = [w_1^T, w_2^T, \ldots, w_s^T]^T$ is divided into $s$ non-overlapping groups, $w_i \in \mathbb{R}^{p_i}, i = 1, 2, \ldots, s, \lambda > 0$ is the
regularization parameter, and

\[
\varpi(W) = \sum_{i=1}^{s} \|w_i\|_q
\]

(2)
is the \( \ell_1/\ell_q \) norm with \( \| \cdot \|_q \) denoting the vector \( \ell_q \) norm \( (q \geq 1) \). The \( \ell_1/\ell_q \)-regularization belongs to the composite absolute penalties (CAP) \cite{38} family. When \( q = 1 \), the problem (1) reduces to the \( \ell_1 \)-regularized problem. When \( q > 1 \), the \( \ell_1/\ell_q \)-regularization facilitates group sparsity in the resulting model, which is desirable in many applications of regression and classification.

The practical challenge in the use of the \( \ell_1/\ell_q \)-regularization lies in the development of efficient algorithms for solving (1), due to the non-smoothness of the \( \ell_1/\ell_q \)-regularization. According to the black-box Complexity Theory \cite{25, 26}, the optimal first-order black-box method \cite{25, 26} for solving the class of nonsmooth convex problems converges as \( O(1/\sqrt{k}) \) \( (k \) denotes the number of iterations), which is slow. Existing algorithms focus on solving the problem (1) or its equivalent constrained version for \( q = 2, \infty \), and they can not be easily extended to the general case. In order to systematically study the practical performance of the \( \ell_1/\ell_q \)-regularization family, it is of great importance to develop efficient algorithms for solving (1) for any \( q \) larger than 1.

1.1 First-Order Methods Applicable for (1)

When treating \( f(\cdot) \) as the general non-smooth convex function, we can apply the subgradient descent \cite{5, 25, 26}:

\[
X_{i+1} = X_i - \gamma_i G_i,
\]

(3)

where \( G_i \in \partial f(X_i) \) is a subgradient of \( f(\cdot) \) at \( X_i \), and \( \gamma_i \) a step size. There are several different types of step size rules, and more details can be found in \cite{5, 25}. Subgradient descent is proven to converge, and it can yield a convergence rate of \( O(1/\sqrt{k}) \) for \( k \) iterations. However, SD has the following two disadvantages: 1) SD converges slowly; and 2) the iterates of SD are very rarely at the points of non-differentiability \cite{7}, thus it might not achieve the desirable sparse solution (which is usually at the point of non-differentiability) within a limited number of iterations.

Coordinate Descent \cite{33} and its recent extension—Coordinate Gradient Descent (CGD) can be applied for optimizing the non-differentiable composite function \cite{34}. Coordinate descent has been applied for the \( \ell_1 \)-norm regularized least squares \cite{9}, \( \ell_1/\ell_\infty \)-norm regularized least squares \cite{16}, and the sparse group Lasso \cite{10}. Coordinate gradient descent has been applied for the group Lasso logistic regression \cite{21}. Convergence results for CD and CGD have been established, when the non-differentiable part is separable \cite{33, 34}. However, there is no global convergence rate for CD and CGD (Note, CGD is reported to have a \textit{local} linear convergence rate under certain conditions \cite[Theorem 4]{34}). In addition, it is not clear whether CD and CGD are applicable for solving the problem (1) with an arbitrary \( q \geq 1 \).

Fixed Point Continuation \cite{12, 31} was recently proposed for solving the \( \ell_1 \)-norm regularized optimization (i.e., \( \varpi(W) = \|W\|_1 \)). It is based on the following fixed
point iteration:

\[ X_{i+1} = \mathcal{P}_{\lambda \tau}^\varpi(X_i - \tau l'(X_i)), \tag{4} \]

where \( \mathcal{P}_{\lambda \tau}^\varpi(W) = \text{sgn}(W) \odot \max(W - \lambda \tau, 0) \) is an operator and \( \tau > 0 \) is the step size. The fixed point iteration (4) can be applied to solve (1) for any convex penalty \( \varpi(W) \), with the operator \( \mathcal{P}_{\lambda \tau}^\varpi(\cdot) \) being defined as:

\[ \mathcal{P}_{\lambda \tau}^\varpi(W) = \arg \min_W \frac{1}{2} \| X - W \|_2^2 + \lambda \tau \varpi(X). \tag{5} \]

The operator \( \mathcal{P}_{\lambda \tau}^\varpi(\cdot) \) is called the proximal operator [13, 22, 36], and is guaranteed to be non-expansive. With a properly chosen \( \tau \), the fixed point iteration (4) can converge to the fixed point \( X^* \) satisfying

\[ X^* = \mathcal{P}_{\lambda \tau}^\varpi(X^* - \tau l'(X^*)). \tag{6} \]

It follows from (5) and (6) that,

\[ 0 \in X^* - (X^* - \tau l'(X^*)) + \lambda \tau \partial \varpi(X^*), \tag{7} \]

which together with \( \tau > 0 \) indicates that \( X^* \) is the optimal solution to (1). In [3, 27], the gradient descent method is extended to optimize the composite function in the form of (1), and the iteration step is similar to (4). The extended gradient descent method is proven to yield the convergence rate of \( O(1/k) \) for \( k \) iterations. However, as pointed out in [3, 27], the scheme in (4) can be further accelerated for solving (1).

Finally, there are various online learning algorithms that have been developed for dealing with large-scale data, e.g., the truncated gradient method [15], the forward-looking subgradient [7], and the regularized dual averaging [35] (which is based on the dual averaging method proposed in [28]). When applying the aforementioned online learning methods for solving (1), a key building block is the operator \( \mathcal{P}_{\lambda \tau}^\varpi(\cdot) \).

### 1.2 Main Contributions

In this paper, we develop an efficient algorithm for solving the \( \ell_1/\ell_q \)-regularized problem (1), for any \( q \geq 1 \). More specifically, we develop the GLEP\(_{\ell_1q}\) algorithm\(^1\) which makes use of the accelerated gradient method [3, 27] for minimizing the composite objective functions. GLEP\(_{\ell_1q}\) has the following two favorable properties: (1) It is applicable to any smooth convex loss \( l(\cdot) \) (e.g., the least squares loss and the logistic loss) and any \( q \geq 1 \). Existing algorithms are mainly focused on \( \ell_1/\ell_2 \)-regularization and/or \( \ell_1/\ell_\infty \)-regularization. To the best of our knowledge, this is the first work that provides an efficient algorithm for solving (1) with any \( q \geq 1 \); and (2) It achieves a global convergence rate of \( O(1/k) \) \((k \) denotes the number of iterations) for the smooth convex loss \( l(\cdot) \). In comparison, although the methods proposed in [1, 6, 16, 29] converge, there is no known convergence rate; and the method proposed in [21] has a local linear convergence rate under certain conditions [34, Theorem 4]. In addition, these methods are not applicable for an arbitrary \( q \geq 1 \).

\(^1\)GLEP\(_{\ell_1q}\) stands for Group Sparsity Learning via the \( \ell_1/\ell_q \)-regularized Euclidean Projection.
The main technical contribution of this paper is the development of an efficient algorithm for computing the \( \ell_1/\ell_q \)-regularized Euclidean projection (EP\(_{1,q}\)), which is a key building block in the proposed GLEP\(_{1,q}\) algorithm. More specifically, we analyze the key theoretical properties of the solution of EP\(_{1,q}\), based on which we develop an efficient algorithm for EP\(_{1,q}\) by solving two zero finding problems. In addition, our theoretical analysis reveals why EP\(_{1,q}\) for the general \( q \) is significantly more challenging than the special cases such as \( q = 2 \). We have conducted experimental studies to demonstrate the efficiency of the proposed algorithm.

### 1.3 Related Work

We briefly review recent studies on \( \ell_1/\ell_q \)-regularization, most of which focus on \( \ell_1/\ell_2 \)-regularization and/or \( \ell_1/\ell_\infty \)-regularization.

- \( \ell_1/\ell_2 \)-Regularization: The group Lasso was proposed in [37] to select the groups of variables for prediction in the least squares regression. In [21], the idea of group lasso was extended for classification by the logistic regression model, and an algorithm via the coordinate gradient descent [34] was developed. In [29], the authors considered joint covariate selection for grouped classification by the logistic loss, and developed a blockwise boosting Lasso algorithm with the boosted Lasso [39]. In [1], the authors proposed to learn the sparse representations shared across multiple tasks, and designed an alternating algorithm. The Spectral projected-gradient (Spg) algorithm was proposed for solving the \( \ell_1/\ell_2 \)-ball constrained smooth optimization problem [4], equipped with an efficient Euclidean projection that has expected linear runtime. The \( \ell_1/\ell_2 \)-regularized multi-task learning was proposed in [18], and the equivalent smooth reformulations were solved by the Nesterov’s method [20].

- \( \ell_1/\ell_\infty \)-Regularization: A blockwise coordinate descent algorithm [33] was developed for the multi-task Lasso [16]. It was applied to the neural semantic basis discovery problem. In [30], the authors considered the multi-task learning via the \( \ell_1/\ell_\infty \)-regularization, and proposed to solve the equivalent \( \ell_1/\ell_\infty \)-ball constrained problem by the projected gradient descent. In [24], the authors considered the multivariate regression via the \( \ell_1/\ell_\infty \)-regularization, showed that the high-dimensional scaling of \( \ell_1/\ell_\infty \)-regularization is qualitatively similar to that of ordinary \( \ell_1 \)-regularization, and revealed that, when the overlap parameter is large enough (\( > 2/3 \)), \( \ell_1/\ell_\infty \)-regularization yields the improved statistical efficiency over \( \ell_1 \)-regularization.

- \( \ell_1/\ell_q \)-Regularization: In [6], the authors studied the problem of boosting with structural sparsity, and developed several boosting algorithms for regularization penalties including \( \ell_1, \ell_\infty, \ell_1/\ell_2, \) and \( \ell_1/\ell_\infty \). In [48], the composite absolute penalties (CAP) family was introduced, and an algorithm called iCAP was developed. iCAP employed the least squares loss and the \( \ell_1/\ell_\infty \) regularization, and was implemented by the boosted Lasso [39]. The multivariate regression with the \( \ell_1/\ell_q \)-regularization was studied in [17]. In [23], a unified framework was provided for establishing consistency and convergence rates for the regularized M-estimators, and the results for \( \ell_1/\ell_q \) regularization was established.
1.4 Notation

Throughout this paper, scalars are denoted by italic letters, and vectors by bold face letters. Let $X, Y, \ldots$ denote the $p$-dimensional parameters, $x_i, y_i, \ldots$ the $p_i$-dimensional parameters of the $i$-th group, and $x_t$ the $i$-th component of $x$. We denote $q = \frac{1}{1+1}$, and thus $q$ and $\bar{q}$ satisfy the following relationship: $\frac{1}{q} + \frac{1}{\bar{q}} = 1$. We use the following componentwise operators: $\odot, | \cdot |$ and $\text{sgn}(\cdot)$. Specifically, $z = x \odot y$ denotes $z_i = x_i y_i$; $y = |x|$ denotes $y_i = |x_i|$; and $y = \text{sgn}(x)$ denotes $y_i = \text{sgn}(x_i)$, where $\text{sgn}(\cdot)$ is the signum function: $\text{sgn}(t) = 1$ if $t > 0$; $\text{sgn}(t) = 0$ if $t = 0$; and $\text{sgn}(t) = -1$ if $t < 0$.

2 The Proposed GLEP$_1q$ Algorithm

In this section, we present the proposed GLEP$_1q$ algorithm for solving (1) in the batch learning setting. The main technical contribution lies in the development of an efficient algorithm for the $\ell_1/\ell_q$-regularized Euclidean projection. Specifically, we analyze the key theoretical properties of the projection in Section 2.1 and show that the projection can be computed by solving two zero finding problems in Section 2.2. Note that, one can develop the online learning algorithm for (1) using the online learning algorithms discussed in the last section, where the $\ell_1/\ell_q$-regularized Euclidean projection is also a key building block.

We first construct the following model for approximating the composite function $\mathcal{M}(\cdot)$ at the point $X$ [3][27]:

$$\mathcal{M}_{L, X}(Y) = \left[\text{loss}(X) + \langle \text{loss}'(X), Y - X \rangle\right] + L \lambda \varpi(Y) + L \frac{1}{2} \| Y - X \|^2_2, \quad (8)$$

where $L > 0$. In the model $\mathcal{M}_{L, X}(Y)$, we apply the first-order Taylor expansion at the point $X$ (including all terms in the square bracket) for the smooth loss function $l(\cdot)$, and directly put the non-smooth penalty $\varpi(\cdot)$ into the model. The regularization term $L \frac{1}{2} \| Y - X \|^2_2$ prevents $Y$ from walking far away from $X$, thus the model can be a good approximation to $f(Y)$ in the neighborhood of $X$.

The accelerated gradient method is based on two sequences $\{X_i\}$ and $\{S_i\}$ in which $\{X_i\}$ is the sequence of approximate solutions, and $\{S_i\}$ is the sequence of search points. The search point $S_i$ is the affine combination of $X_{i-1}$ and $X_i$ as

$$S_i = X_i + \beta_i (X_i - X_{i-1}), \quad (9)$$

where $\beta_i$ is a properly chosen coefficient. The approximate solution $X_{i+1}$ is computed as the minimizer of $\mathcal{M}_{L, S_i}(Y)$:

$$X_{i+1} = \arg \min_Y \mathcal{M}_{L, S_i}(Y), \quad (10)$$

where $L_i$ is determined by line search, e.g., the Armijo-Goldstein rule so that $L_i$ should be appropriate for $S_i$.

The algorithm for solving (1) is presented in Algorithm[1]. GLEP$_1q$ inherits the optimal convergence rate of $O(1/k^2)$ from the accelerated gradient method. In Algorithm[1] a key subroutine is (10), which can be computed as $X_{i+1} = \pi_{1q}(S_i - \ldots$
Algorithm 1 GLEP$_{1,q}$: Group Sparsity Learning via the $\ell_1/\ell_q$-regularized Euclidean Projection

**Input:** $\lambda_1 \geq 0, \lambda_2 \geq 0, L_0 > 0, X_0, k$

**Output:** $X_{k+1}$

1. Initialize $X_1 = X_0$, $\alpha_{-1} = 0, \alpha_0 = 1$, and $L = L_0$.
2. for $i = 1$ to $k$ do
   3. Set $\beta_i = \frac{\alpha_i - 1}{\alpha_{i-1}}$, $S_i = X_i + \beta_i(X_i - X_{i-1})$
   4. Find the smallest $L = L_{i-1}, 2L_{i-1}, \ldots$ such that $f(X_{i+1}) \leq \mathcal{M}_{L,S_i}(X_{i+1})$,
      where $X_{i+1} = \text{arg min}_{Y} \mathcal{M}_{L,S_i}(Y)$
   5. Set $L_i = L$ and $\alpha_{i+1} = \frac{1 + \sqrt{1 + 4\alpha_i^2}}{2}$
   6. end for

$l'(S_i)/L_i, \lambda/L_i$, where $\pi_{1,q}(\cdot)$ is the $\ell_1/\ell_q$-regularized Euclidean projection (EP$_{1,q}$) problem:

$$
\pi_{1,q}(V, \lambda) = \text{arg min}_{X \in \mathbb{R}^p} \frac{1}{2}\|X - V\|_2^2 + \lambda \sum_{i=1}^{s} \|x_i\|_q.
$$

The efficient computation of (11) for any $q > 1$ is the main technical contribution of this paper. Note that the $s$ groups in (11) are independent. Thus the optimization in (11) decouples into a set of $s$ independent $\ell_q$-regularized Euclidean projection problems:

$$
\pi_q(v) = \text{arg min}_{x \in \mathbb{R}^n} \left( g(x) = \frac{1}{2}\|x - v\|_2^2 + \lambda \|x\|_q \right),
$$

where $n = p_i$ for the $i$-th group. Next, we study the key properties of (12).

2.1 Properties of the Optimal Solution to (12)

The function $g(\cdot)$ is strictly convex, and thus it has a unique minimizer, as summarized below:

**Lemma 1** The problem (12) has a unique minimizer.

Next, we show that the optimal solution to (12) is given by zero under a certain condition, as summarized in the following theorem:

**Theorem 1** $\pi_q(v) = 0$ if and only if $\lambda \geq \|v\|_{\bar{q}}$.

**Proof:** Let us first compute the directional derivative of $g(x)$ at the point $0$:

$$
Dg(0)[u] = \lim_{\alpha \to 0} \frac{1}{\alpha} [g(\alpha u) - g(0)] = -\langle v, u \rangle + \lambda \|u\|_q,
$$

where $u$ is a given direction. According to the Hölder’s inequality, we have

$$
|\langle u, v \rangle| \leq \|u\|_q \|v\|_{\bar{q}}, \forall u.
$$
Therefore, we have

$$Dg(0)[u] \geq 0, \forall u,$$

if and only if $$\lambda \geq \|v\|_q$$. The result follows, since (13) is the necessary and sufficient condition for 0 to be the optimal solution of (12).

Next, we focus on solving (12) for $$1 < \lambda < \|v\|_q$$. We first consider solving (12) in the case of $$1 < q < \infty$$, which is the main technical contribution of this paper. We begin with a lemma that summarizes the key properties of the optimal solution to the problem (12):

**Lemma 2** Let $$1 < q < \infty$$ and $$0 < \lambda < \|v\|_q$$. Then, $$x^*$$ is the optimal solution to the problem (12) if and only if it satisfies:

$$x^* + \lambda \|x^*\|_q^{1-q}x^{(q-1)} = v,$$  (14)

where $$y \equiv x^{(q-1)}$$ is defined component-wisely as: $$y_i = \text{sgn}(x_i)|x_i|^{q-1}$$. Moreover, we have

$$\pi_q(v) = \text{sgn}(v) \oplus \pi_q(|v|),$$  (15)

$$\text{sgn}(x^*) = \text{sgn}(v),$$  (16)

$$0 < |x^*_i| < |v_i|, \forall i \in \{i | v_i \neq 0\}. $$  (17)

**Proof:** Since $$\lambda < \|v\|_q$$, it follows from Theorem 1 that the optimal solution $$x^* \neq 0$$.

$$\|x\|_q$$ is differentiable when $$x \neq 0$$, so is $$g(x)$$. Therefore, the sufficient and necessary condition for $$x^*$$ to be the solution of (12) is $$g'(x^*) = 0$$, i.e., (14). Denote $$c^* \equiv \lambda \|x^*\|_q^{1-q} > 0$$. It follows from (14) that (15) holds, and

$$\text{sgn}(x^*_i) \left(|x^*_i| + c^*|x^*_i|^{q-1}\right) = v_i,$$  (18)

from which we can verify (16) and (17).

It follows from Lemma 2 that i) if $$v_i = 0$$ then $$x^*_i = 0$$; and ii) $$\pi_q(v)$$ can be easily obtained from $$\pi_q(|v|)$$. Thus, we can restrict our following discussion to $$v > 0$$. 

Figure 1: Illustration of the failure of the fixed point iteration $$x = v - \lambda \|x\|_q^{1-q}x^{(q-1)}$$ for solving (12). We set $$v = [1, 3]^T$$ and the starting point $$x = [1, 3]^T$$. The vertical axis denotes the values of $$x_1$$ during the iterations.
i.e., \( v_i > 0, \forall i \). It is clear that, the analysis can be easily extended to the general \( v \).

The optimality condition in [14] indicates that \( x^* \) might be solved via the fixed point iteration

\[
x = \eta(x) \equiv v - \lambda \|x\|_q^{1-q} x^{(q-1)},
\]

which is, however, not guaranteed to converge (see Figure [1] for examples), as \( \eta(\cdot) \) is not necessarily a contraction mapping [14, Proposition 3]. In addition, \( x^* \) cannot be trivially solved by firstly guessing \( c = \|x\|_q^{1-q} \) and then finding the root of \( x + \lambda c x^{(q-1)} = v \), as when \( c \) increases, the values of \( x \) obtained from \( x + \lambda c x^{(q-1)} = v \) decrease, so that \( c = \|x\|_q^{1-q} \) increases as well (note that \( 1 - q < 0 \)).

### 2.2 Computing the Optimal Solution \( x^* \) by Zero Finding

In the following, we show that \( x^* \) can be obtained by solving two zero finding problems. Below, we construct our first auxiliary function \( h_c^v(\cdot) \) and reveal its properties:

**Definition 1 (Auxiliary Function \( h_c^v(\cdot) \))** Let \( c > 0, 1 < q < \infty, \) and \( v > 0 \). We define the auxiliary function \( h_c^v(\cdot) \) as follows:

\[
h_c^v(x) = x + cx^{q-1} - v, 0 \leq x \leq v. \tag{19}
\]

**Lemma 3** Let \( c > 0, 1 < q < \infty, \) and \( v > 0 \). Then, \( h_c^v(\cdot) \) has a unique root in the interval \((0, v)\).

**Proof:** It is clear that \( h_c^v(\cdot) \) is continuous and strictly increasing in the interval \([0, v]\), \( h_c^v(0) = -v < 0 \), and \( h_c^v(v) = cv^{q-1} > 0 \). According to the Intermediate Value Theorem, \( h_c^v(\cdot) \) has a unique root lying in the interval \((0, v)\). This concludes the proof. \( \square \)

**Corollary 1** Let \( x, v \in \mathbb{R}^n \), \( c > 0, 1 < p < \infty, \) and \( v > 0 \). Then, the function

\[
\varphi_c^v(x) = x + cx^{(q-1)} - v, 0 \leq x \leq v \tag{20}
\]

has a unique root.

Let \( x^* \) be the optimal solution satisfying (14). Denote \( c^* = \lambda \|x^*\|_q^{1-q} \). It follows from Lemma 2 and Corollary 1 that \( x^* \) is the unique root of \( \varphi_{c^*}^v(\cdot) \) defined in (20), provided that the optimal \( c^* \) is known. Our methodology for computing \( x^* \) is to first compute the optimal \( c^* \) and then compute \( x^* \) by computing the root of \( \varphi_{c^*}^v(\cdot) \). Next, we show how to compute the optimal \( c^* \) by solving a single variable zero finding problem. We need our second auxiliary function \( \omega(\cdot) \) defined as follows:

**Definition 2 (Auxiliary Function \( \omega(\cdot) \))** Let \( 1 < q < \infty \) and \( v > 0 \). We define the auxiliary function \( \omega(\cdot) \) as follows:

\[
c = \omega(x) = (v - x)/x^{q-1}, 0 < x \leq v. \tag{21}
\]

**Lemma 4** In the interval \((0, v)\), \( c = \omega(x) \) is i) continuously differentiable, ii) strictly decreasing, and iii) invertible. Moreover, in the domain \([0, \infty)\), the inverse function \( x = \omega^{-1}(c) \) is continuously differentiable and strictly decreasing.
Definition 3 (Auxiliary Function \( \phi(\cdot) \)) Let \( 1 < q < \infty \), \( 0 < \lambda < \|v\|_{\overline{q}} \), and \( v > 0 \). We define the auxiliary function \( \phi(\cdot) \) as follows:

\[
\phi(c) = \lambda \psi(c) - c, \quad c \geq 0,
\]

where

\[
\psi(c) = \left( \sum_{i=1}^{n} (\omega_{i}^{-1}(c))^q \right)^{\frac{1}{q-1}},
\]

and \( \omega_{i}^{-1}(c) \) is the inverse function of

\[
\omega_{i}(x) = (v_{i} - x)/x^{q-1}, \quad 0 < x \leq v_{i}.
\]

Proof: It is easy to verify that, in the interval \( (0, v] \), \( c = \omega(x) \) is continuously differentiable with a non-positive gradient, i.e., \( \omega'(x) < 0 \). Therefore, the results follow from the Inverse Function Theorem.

It follows from Lemma 4 that given the optimal \( c^* \) and \( v \), the optimal \( x^* \) can be computed via the inverse function \( \omega^{-1}(\cdot) \), i.e., we can represent \( x^* \) as a function of \( c^* \). Since \( \lambda\|x^*\|_{\overline{q}}^{-q} - c^* = 0 \) by the definition of \( c^* \), the optimal \( c^* \) is a root of our third auxiliary function \( \phi(\cdot) \) defined as follows:

\[
\phi(0) = \lambda\|v\|_{\overline{q}}^{-q} > 0, \quad \phi(\overline{\tau}) \leq 0,
\]

where

\[
\overline{\tau} = \max_{i} c_{i},
\]

\[
c_{i} = \omega_{i}(v_{i}\epsilon), \quad i = 1, 2, \ldots, n.
\]

Proof: From Lemma 4, the function \( \omega_{i}^{-1}(c) \) is continuously differentiable in \( [0, \infty) \). It is easy to verify that \( \omega_{i}^{-1}(c) > 0, \forall c \in [0, \infty) \). Thus, \( \phi(\cdot) \) in (22) is continuously differentiable in \( [0, \infty) \).

It is clear that \( \phi(0) = \lambda\|v\|_{\overline{q}}^{-q} > 0 \). Next, we show \( \phi(\overline{\tau}) \leq 0 \). Since \( 0 < \lambda < \|v\|_{\overline{q}} \), we have

\[
0 < \epsilon < 1.
\]

It follows from (24), (26), (27) and (28) that \( 0 < c_{i} \leq \overline{\tau}, \forall i \). Let \( x = [x_{1}, x_{2}, \ldots, x_{n}]^{T} \) be the root of \( \omega_{i}^{-1}(\cdot) \) (see Corollary 1). Then, \( x_{i} = \omega_{i}^{-1}(\overline{\tau}) \). Since \( \omega_{i}^{-1}(\cdot) \) is strictly decreasing (see Lemma 4), \( c_{i} \leq \overline{\tau}, v_{i}\epsilon = \omega_{i}^{-1}(c_{i}) \), and \( x_{i} = \omega_{i}^{-1}(\overline{\tau}) \), we have

\[
x_{i} \leq v_{i}\epsilon.
\]
Combining (24), (29), and \( \psi = \omega_i(x_i) \), we have \( \psi \geq v_i(1 - \epsilon)/x_i^{q-1} \), since \( \omega_i(\cdot) \) is strictly decreasing. It follows that \( x_i \geq \left( \frac{v_i(1 - \epsilon)}{\psi} \right)^{\frac{1}{q-1}} \). Thus, the following holds:

\[
\psi(\psi) = \left( \sum_{i=1}^{n} (\omega_i^{-1}(\psi))^{q} \right)^{\frac{1}{q}} = \left( \sum_{i=1}^{n} x_i^{q} \right)^{\frac{1}{q}} \leq \frac{\psi}{\| v \|^{1/2 - 1}},
\]

which leads to

\[
\phi(\psi) = \lambda \psi(\psi) - \psi \leq \psi \left( \frac{\lambda}{\| v \|^{1/2 - 1}} - 1 \right) = 0,
\]

where the last equality follows from (25).

**Corollary 2** Let \( 1 < q < \infty, 0 < \lambda < \| v \|^{-1/2}, \) \( v > 0, \) and \( \underline{c} = \min_i c_i, \) where \( c_i \) ’s are defined in (27). We have \( 0 < \underline{c} \leq \underline{\psi} \) and \( \phi(\underline{\psi}) \geq 0. \)

Following Lemma 3 and Corollary 2, we can find at least one root of \( \phi(\cdot) \) in the interval \([\underline{c}, \underline{\psi}]\). In the following theorem, we show that \( \phi(\cdot) \) has a unique root:

**Theorem 2** Let \( 1 < q < \infty, 0 < \lambda < \| v \|^{-1/2}, \) \( v > 0. \) Then, in \([\underline{c}, \underline{\psi}]\), \( \phi(\cdot) \) has a unique root, denoted by \( \psi^* \), and the root of \( \phi_{\psi^*}(\cdot) \) is the optimal solution to (12).

**Proof:** From Lemma 3 and Corollary 2, we have \( \phi(\underline{\psi}) \leq 0 \) and \( \phi(\underline{c}) \geq 0. \) If either \( \phi(\underline{\psi}) = 0 \) or \( \phi(\underline{c}) = 0, \) \( \underline{\psi} \) or \( \underline{c} \) is a root of \( \phi(\cdot) \). Otherwise, we have \( \phi(\underline{c}) \phi(\underline{\psi}) < 0. \) As \( \phi(\cdot) \) is continuous in \([0, \infty)\), we conclude that \( \phi(\cdot) \) has a root in \((\underline{c}, \underline{\psi})\) according to the Intermediate Value Theorem.

Next, we show that \( \phi(\cdot) \) has a unique root in the interval \([0, \infty)\). We prove this by contradiction. Assume that \( \phi(\cdot) \) has two roots: \( 0 < c_1 < c_2. \) From Corollary 1, \( \phi_{c_1}(\cdot) \) and \( \phi_{c_2}(\cdot) \) have unique roots. Denote \( x^1 = [x_1^1, x_2^1, \ldots, x_n^1]^T \) and \( x^2 = [x_1^2, x_2^2, \ldots, x_n^2]^T \) as the roots of \( \phi_{c_1}(\cdot) \) and \( \phi_{c_2}(\cdot) \), respectively. We have \( 0 < x_i^1, x_i^2 < v_i, \forall i. \) It follows from (22-24) that

\[
\begin{align*}
\text{x}^1 + \lambda \| \text{x}^1 \|^{1-q} \text{x}^1(1-q^{-1}) - \text{v} &= 0, \\
\text{x}^2 + \lambda \| \text{x}^2 \|^{1-q} \text{x}^2(1-q^{-1}) - \text{v} &= 0.
\end{align*}
\]

According to Lemma 3 \( x^1 \) and \( x^2 \) are the optimal solution of (12). From Lemma 1, we have \( x^1 = x^2 \). However, since \( x_i^1 = \omega_i^{-1}(c_1), x_i^2 = \omega_i^{-1}(c_2), \omega_i^{-1}(\cdot) \) is a strictly decreasing function in \([0, \infty)\) by Lemma 3 and \( c_1 < c_2, \) we have \( x_i^1 > x_i^2, \forall i. \) This leads to a contradiction. Therefore, we conclude that \( \phi(\cdot) \) has a unique root in \([\underline{c}, \underline{\psi}]\).

From the above arguments, it is clear that, the root of \( \phi_{\psi^*}(\cdot) \) is the optimal solution to (12).

**Remark 1** When \( q = 2, \) we have \( \underline{c} = \underline{\psi} = \frac{\lambda}{\| v \|^{-1/2}}. \) It is easy to verify that \( \phi(\underline{\psi}) = \phi(\underline{\psi}) = 0 \) and

\[
\pi_2(\text{v}) = \frac{\| \text{v} \|_2 - \lambda}{\| \text{v} \|_2} \text{v}.
\]

Therefore, when \( q = 2, \) we obtain a closed-form solution.
2.3 Solving the Zero Finding Problem by Bisection

Let $1 < q < \infty$, $0 < \lambda < \|v\|_q$, $v > 0$, $\overline{\tau} = \max_i v_i$, $\underline{\nu} = \min_i v_i$, and $\delta > 0$ be a small constant (e.g., $\delta = 10^{-8}$ in our experiments). When $q > 2$, we have

$$\underline{c} = \frac{1 - \epsilon}{e^{q-1}v^q - 2} \quad \text{and} \quad \overline{c} = \frac{1 - \epsilon}{e^{q-1}v^q - 2}. $$

When $1 < q < 2$, we have

$$\underline{c} = \frac{1 - \epsilon}{e^{q-1}v^q - 2} \quad \text{and} \quad \overline{c} = \frac{1 - \epsilon}{e^{q-1}v^q - 2}. $$

If either $\phi(\tau) = 0$ or $\phi(\underline{c}) = 0$, $\tau$ is the unique root of $\phi(\cdot)$, or $\underline{c}$ is the unique root of $\phi(\cdot)$. Otherwise, we can find the unique root of $\phi(\cdot)$ by bisection in the interval $[\underline{c}, \overline{c}]$, which costs at most

$$N = \log_2 \frac{(1 - \epsilon)|\overline{\tau}^q - \underline{\nu}^q|}{e^{q-1}|\overline{\tau}^q - \underline{\nu}^q - 2\delta}$$

iterations for achieving an accuracy of $\delta$. Let $[c_1, c_2]$ be the current interval of uncertainty, and we have computed $\omega_i^{-1}(c_1)$ and $\omega_i^{-1}(c_2)$ in the previous bisection iterations. Setting $c = \frac{c_1 + c_2}{2}$, we need to evaluate $\phi(c)$ by computing $\omega_i^{-1}(c_2)$. Setting $c = \frac{c_1 + c_2}{2}$, we need to evaluate $\phi(c)$ by computing $\omega_i^{-1}(c_2)$.

It is easy to verify that $\omega_i^{-1}(c)$ is the root of $h^1_i(\cdot)$ in the interval $(0, v_i)$. Since $\omega_i^{-1}(\cdot)$ is a strictly decreasing function (see Lemma 4), the following holds:

$$\omega_i^{-1}(c_2) < \omega_i^{-1}(c) < \omega_i^{-1}(c_1),$$

and thus $\omega_i^{-1}(c)$ can be solved by bisection using at most

$$\log_2 \frac{\omega_i^{-1}(c_2) - \omega_i^{-1}(c_1)}{\delta} < \log_2 \frac{v_i}{\delta} \leq \log_2 \frac{\overline{\tau}}{\delta}$$

iterations for achieving an accuracy of $\delta$. For given $v$, $\lambda$, and $\delta$, $N$ and $\overline{\tau}$ are constant, and thus it costs $O(n)$ for finding the root of $\phi(\cdot)$. Once $c^*$, the root of $\phi(\cdot)$ is found, it costs $O(n)$ flops to compute $x^*$ as the unique root of $\varphi^*_x(\cdot)$. Therefore, the overall time complexity for solving (12) is $O(n)$.

We have shown how to solve (12) for $1 < q < \infty$. For $q = 1$, the problem (12) is reduced to the one used in the standard Lasso, and it has the following closed-form solution (3):

$$\pi_1(v) = \text{sgn}(v) \odot \max(|v| - \lambda, 0). \quad (31)$$

For $q = \infty$, the problem (12) can computed via (31), as summarized in the following theorem:

**Theorem 3** Let $q = \infty$, $\bar{q} = 1$, and $0 < \lambda < \|v\|_{\bar{q}}$. Then we have

$$\pi_\infty(v) = \text{sgn}(v) \odot \min(|v|, t^*), \quad (32)$$

where $t^*$ is the unique root of

$$h(t) = \sum_{i=1}^n \max(|v_i| - t, 0) - \lambda. \quad (33)$$
Proof: Making use of the property that $\|x\|_\infty = \max_{\|y\|_1 \leq 1} \langle y, x \rangle$, we can rewrite (12) in the case of $q = \infty$ as

$$\min_x \max_{y:\|y\|_1 \leq \lambda} s(x, y) \equiv \frac{1}{2} \|x - v\|_2^2 + \langle y, x \rangle. \quad (34)$$

The function $s(x, y)$ is continuously differentiable in both $x$ and $y$, convex in $x$ and concave in $y$, and the feasible domains are solids. According to the well-known von Neumann Lemma [25], the min-max problem (34) has a saddle point, and thus the minimization and maximization can be exchanged. Setting the derivative of $s(x, y)$ with respect to $x$ to zero, we have

$$x = v - y. \quad (35)$$

Thus we obtain the following problem:

$$\min_{y:\|y\|_1 \leq \lambda} \frac{1}{2} \|y - v\|_2^2, \quad (36)$$

which is the problem of the Euclidean projection onto the $\ell_1$ ball [4, 6, 20]. It has been shown that the optimal solution $y^*$ to (36) for $\lambda < \|v\|_1$ can be obtained by first computing $t^*$ as the unique root of (33) in linear time, and then computing $y^*$ as

$$y^* = \text{sgn}(v) \odot \max(|v| - t^*, 0). \quad (37)$$

It follows from (35) and (37) that (32) holds. □

We conclude this section by summarizing the main steps for solving the $\ell_q$-regularized Euclidean projection in Algorithm 2.

3 Experiments

We have conducted experiments to evaluate the efficiency of the proposed algorithm using both synthetic and real-world data. We set the regularization parameter as $\lambda = r \times \lambda_{\text{max}}^q$, where $0 < r \leq 1$ is the ratio, and $\lambda_{\text{max}}^q$ is the maximal value above which the $\ell_1/\ell_q$-norm regularized problem (1) obtains a zero solution (see Theorem 1). We try the following values for $q$: 1.25, 1.5, 1.75, 2, 2.33, 3, 5, and $\infty$. The source codes, included in the SLEP package [19], are available online.[3]

3.1 Simulation Studies

We use the synthetic data to study the effectiveness of the $\ell_1/\ell_q$-norm regularization for reconstructing the jointly sparse matrix under different values of $q > 1$. Let $A \in \mathbb{R}^{m \times d}$ be a measurement matrix with entries being generated randomly from the standard normal distribution, $X^* \in \mathbb{R}^{d \times k}$ be the jointly sparse matrix with the first $\tilde{d} < d$ rows being nonzero and the remaining rows exactly zero, $Y = AX^* + Z$ be the response matrix, and $Z \in \mathbb{R}^{m \times k}$ is the noise matrix whose entries are drawn randomly from the
Algorithm 2: $\ell_q$-regularized Euclidean projection

**Input:** $\lambda > 0$, $q \geq 1$, $v \in \mathbb{R}^n$

**Output:** $x^* = \pi_q(v) = \arg \min_{x \in \mathbb{R}^n} \frac{1}{2} \|x - v\|_2^2 + \lambda \|x\|_q$

1. Compute $\bar{q} = \frac{q}{q - 1}$
2. if $\|v\|_{\bar{q}} \leq \lambda$ then
   3. Set $x^* = 0$, return
4. end if
5. if $q = 1$ then
   6. Set $x^* = \text{sgn}(v) \odot \max(|v| - \lambda, 0)$
7. else if $q = 2$ then
   8. Set $x^* = \frac{\|v\|_2 - \lambda}{\|v\|_2} v$
9. else if $q = \infty$ then
   10. Obtain $t^*$, the unique root of $h(t)$, via the improved bisection method
   11. Set $x^* = \text{sgn}(v) \odot \min(|v|, t^*)$
12. else
   13. Compute $c^*$, the unique root of $\phi(c)$, via bisection in the interval $[c, \bar{c}]$ (Theorem 2)
   14. Obtain $x^*$ as the unique root of $\varphi^{\nu}_{c^*}(\cdot)$
15. end if

normal distribution with mean zero and standard deviation $\sigma = 0.1$. We treat each row of $X^*$ as a group, and estimate $X^*$ from $A$ and $Y$ by solving the following $\ell_1/\ell_q$-norm regularized problem:

$$X = \arg \min_W \frac{1}{2} \|AW - Y\|_F^2 + \lambda \sum_{i=1}^d \|W_i\|_q,$$

where $W^i$ denotes the $i$-th row of $W$. We set $m = 100$, $d = 200$, and $d = k = 50$. We try two different settings for $X^*$, by drawing its nonzero entries randomly from 1) the uniform distribution in the interval $[0, 1]$ and 2) the standard normal distribution.

We compute the solutions corresponding to a sequence of decreasing values of $\lambda = r \times \lambda_{\text{max}}^q$, where $r = 0.9^{i-1}$, for $i = 1, 2, \ldots, 100$. In addition, we use the solution corresponding to the $0.9^i \times \lambda_{\text{max}}^q$ as the “warm” start for $0.9^{i+1} \times \lambda_{\text{max}}^q$. We report the results in Figure 2 from which we can observe: 1) the distance between the solution $X$ and the truth $X^*$ usually decreases with decreasing values of $\lambda$; 2) for the uniform distribution (see the plots in the first row), $q = 1.5$ performs the best; 3) for the normal distribution (see the plots in the second row), $q = 1.5, 1.75, 2$ and 3 achieve comparable performance and perform better than $q = 1.25, 5$ and $\infty$; 4) with a properly chosen threshold, the support of $X^*$ can be exactly recovered by the $\ell_1/\ell_q$-norm regularization with an appropriate value of $q$, e.g., $q = 1.5$ for the uniform distribution, and $q = 2$ for the normal distribution; and 5) the recovery of $X^*$ with nonzero entries drawn from the normal distribution is easier than that with entries generated from the uniform distribution.

The existing theoretical results [17, 23] can not tell which $q$ is the best; and we believe that the optimal $q$ depends on the distribution of $X^*$, as indicated from the above
Figure 2: Performance of the $\ell_1/\ell_q$-norm regularization for reconstructing the jointly sparse $X^*$. The nonzero entries of $X^*$ are drawn randomly from the uniform distribution for the plots in the first row, and from the normal distribution for the plots in the second row. Plots in the first two rows show $\|X - X^*\|_F$, the Frobenius norm difference between the solution and the truth; and plots in the third row show the $\ell_2$-norm of each row of the solution $X$. 
results. Therefore, it is necessary to conduct the distribution-specific theoretical studies (note that the previous studies usually make no assumption on $X^*$). The proposed GLEP$_{1q}$ algorithm shall help verify the theoretical results to be established.

3.2 Performance on the Letter Data Set

We apply the proposed GLEP$_{1q}$ algorithm for multi-task learning on the Letter data set [29], which consists of 45,679 samples from 8 default tasks of two-class classification problems for the handwritten letters: c/e, g/y, m/n, a/g, i/j, a/o, f/t, h/n. The writings were collected from over 180 different writers, with the letters being represented by $8 \times 16$ binary pixel images. We use the least squares loss for $l(\cdot)$.

3.2.1 Efficiency Comparison with Spg

We compare GLEP$_{1q}$ with the Spg algorithm proposed in [4]. Spg is a specialized solver for the $\ell_1/\ell_2$-ball constrained optimization problem, and has been shown to outperform existing algorithms based on blockwise coordinate descent and projected gradient. In Figure 3 we report the computational time under different values of $m$ (the number of samples) and $\lambda = r \times \lambda_{\text{max}}^q \, (q = 2)$. It is clear from the plots that GLEP$_{1q}$ is much more efficient than Spg, which may attribute to: 1) GLEP$_{1q}$ has a
better convergence rate than Spg; and 2) when $q = 2$, the EP$_{1q}$ in GLEP$_{1q}$ can be computed analytically (see Remark 1), while this is not the case in Spg.

3.2.2 Efficiency under Different Values of $q$

We report the computational time (seconds) of GLEP$_{1q}$ under different values of $q$, $\lambda = r \times \lambda_{\text{max}}^q$ and $m$ (the number of samples) in Figure 4. We can observe from this figure that the computational time of GLEP$_{1q}$ under different values of $q$ (for fixed $r$ and $m$) is comparable. Together with the result on the comparison with Spg for $q = 2$, this experiment shows the promise of GLEP$_{1q}$ for solving large-scale problems for any $q \geq 1$.

3.2.3 Performance under Different Values of $q$

We randomly divide the Letter data into three non-overlapping sets: training, validation, and testing. We train the model using the training set, and tune the regularization parameter $\lambda = r \times \lambda_{\text{max}}^q$ on the validation set, where $r$ is chosen from \{10$^{-1}$, 5 × 10$^{-2}$, 2 × 10$^{-2}$, 1 × 10$^{-2}$, 5 × 10$^{-3}$, 2 × 10$^{-3}$, 1 × 10$^{-3}$\}. On the testing set, we compute the balanced error rate [11]. We report the results averaged over 10 runs in Figure 5. The title of each plot indicates the percentages of samples used for
training, validation, and testing. The results show that, on this data set, a smaller value of $q$ achieves better performance.

4 Conclusion

In this paper, we propose the GLEP$_{1q}$ algorithm for solving the $\ell_1/\ell_q$-norm regularized problem, for any $q \geq 1$. The main technical contribution of this paper is the efficient algorithm for the $\ell_1/\ell_q$-norm regularized Euclidean projection (EP$_{1q}$), which is a key building block of GLEP$_{1q}$. Specifically, we analyze the key theoretical properties of the solution of EP$_{1q}$, based on which we develop an efficient algorithm for EP$_{1q}$ by solving two zero finding problems. Our analysis also reveals why EP$_{1q}$ for the general $q$ is significantly more challenging than the special cases such as $q = 2$.

In this paper, we focus on the efficient implementation of the $\ell_1/\ell_q$-regularized problem. We plan to study the effectiveness of the $\ell_1/\ell_q$ regularization under different values of $q$ for real-world applications in computer vision and bioinformatics. We also plan to conduct the distribution-specific [8] theoretical studies for different values of $q$.

References


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